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Weakly nonlinear analysis of the secondary bimodal instability in planar nematic convection

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We present a weakly nonlinear analysis of the secondary transition from one- to two-mode structures in planar, extended geometry nematic convection. The secondary growth rate being then an explicit function of the nonlinear coupling terms of the basic equations, we can perform a systematic investigation of the physical origin of the secondary mode. This mode turns out to be selected by a nonlinear homogeneous rotation of the director in the horizontal plane. [S1063-651X(97)51508-X]

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A large class of extended dynamical systems, modeled by nonlinear partial differential ''basic'' equations, presents a progressive transition to spatiotemporal complexity. There usually exists a cascade of instabilities between more and more complex solutions of the basic equations. These solutions, and their stability, are mostly studied by numerical methods [1], that are very accurate but do not allow a direct understanding of the physical mechanisms involved. Slightly above the onset of the first instability, in the low amplitude regime, the weakly nonlinear (WNL) methods [2] are an alternative approach whereby the nonlinear terms of the equations are treated as perturbations. These methods are less accurate, but they often give qualitatively correct results. An additional advantage is that secondary growth rates, instead of being the fully numerical result of an eigenvalue problem, are explicit functions of the nonlinear terms of the basic equations. As we wish to show here using one model system, this allows a *systematic investigation of nonlinear physical mechanisms*. We will consider the transition from monomode to bimodal structures that we found in recent experiments [8]. This transition is often one of the first symmetry breakings on the route to ''complexity,'' but is yet poorly understood.

Our model system consists of the convection instabilities of nematic liquid crystals, which present secondary bifurcations even in the low amplitude regime [3]. Nematics possess two geometric degrees of freedom, associated with the director field $\hat{\mathbf{n}}$ (mean orientation of the elongated molecules), which couple with the others fields of the fluid. In a planar nematic layer, where $\hat{\mathbf{n}}$ is set to a horizontal direction $\hat{\mathbf{x}}$ at the horizontal boundary plates, two linear instability mechanisms exist: electroconvection under the application of an electric voltage [4], and anisotropic thermoconvection under the application of a vertical thermal gradient [5]. Normal rolls (NR), of horizontal wave vector $\mathbf{q}_c$ parallel to $\hat{\mathbf{x}}$ ($\mathbf{q}_c = q_c\hat{\mathbf{x}}$), usually appear at threshold in extended geometry thermoconvection and electroconvection. When the reduced control parameter ϵ, which measures the distance from the convection threshold, is increased, these NR generally undergo a long-wavelength secondary instability that leads to oblique rolls (OR), of wave vector $\mathbf{q} = q_x\hat{\mathbf{x}} + q_y\hat{\mathbf{y}}$ with nonzero q_y [6,8]. In this process the reflection $S:\hat{\mathbf{y}} \mapsto -\hat{\mathbf{y}}$ has been broken, and two variants of OR, $\mathbf{q}$ (''zig,'' such that $q_x q_y > 0$) and $S(\mathbf{q})$ (''zag,'' such that $q_x q_y < 0$) exist. Further increasing ϵ, there occurs a transition to *bimodal varicose* (BV) structures, i.e., the branching of a *dual* mode $\mathbf{k} \neq S(\mathbf{q})$ off the primary OR mode $\mathbf{q}$ [7,8]. Whereas the existence of a short-wavelength destabilization mode of the OR has been confirmed by numerical stability analyzes both in electroconvection and thermoconvection, starting from the basic equations [9,10], no mechanism for this bimodal has been proposed so far. In the following, we will first present a general WNL model for the bimodal secondary instability, and then focus on thermoconvection.

The basic evolution equations for convection instabilities take the general form [10]

$$D\partial_t V = L_\epsilon V + N_2(V,V) + N_3(V,V,V) + \cdots, \quad (1)$$

where V is the local state vector of the fluid ($V=0$ at rest), D, L_ϵ are linear and N_2, N_3 nonlinear differential operators. The WNL methods use as a basic ansatz for approximate solutions of Eq. (1) some eigenmodes $V_1(\mathbf{p})$ of $D^{-1}L_\epsilon$ characterized by their horizontal wave vector $\mathbf{p}$, and their growth rate $\sigma(\mathbf{p};\epsilon)$. In the WNL domain, $\sigma(\mathbf{p};\epsilon) \simeq \tau_{\mathbf{p}}^{-1}[\epsilon - \epsilon_0(\mathbf{p})]$, where $\epsilon_0(\mathbf{p})$ is the linear threshold of the mode $\mathbf{p}$ and $\tau_{\mathbf{p}}$ is a characteristic time; for the critical mode $\mathbf{q}_c$ one has $\epsilon_0(\mathbf{q}_c) = 0$ and $\tau_{\mathbf{q}_c} = 1$. In nematics, experimental investigations [6,8] as well as theoretical studies (see the Fig. 6.7.b of [3]) have shown that the long-wavelength instabilities do not affect the OR, as soon as their wave vector has a sufficiently large q_y component. Hereafter we will use for our investigation of short-wavelength instabilities a primary mode *zig* $\mathbf{q}$ deduced from the thermoconvection experiments $\mathbf{q} = \mathbf{q}_c + \hat{\mathbf{y}}(q_c \tan 10^\circ)$ [8].

The homogeneous BV solutions of Eq. (1) are expressed as

$$V = AV_1(\mathbf{q}) + BV_1(\mathbf{k}) + \text{c.c.} + (\text{second harmonics}). \quad (2)$$

If $\mathbf{k}$ and $\mathbf{q}$ are active, i.e., of small growth rate, their second harmonics, which lie in the subspace of very negative growth rate linear modes, can be obtained by adiabatic elimination:

$$V_2(\mathbf{p},\mathbf{p}') = -L_0^{-1}[N_2[V_1(\mathbf{p}),V_1(\mathbf{p}')] + \text{perm}] \quad \text{for } \mathbf{p} \neq \mathbf{p}'. \quad (3)$$

By inserting Eq. (2) in Eq. (1), and projecting, with an adequate scalar product in the V space, onto the adjoint vectors $U_1(\mathbf{q}),U_1(\mathbf{k})$ defined as in [10], one obtains the amplitude equations

$$\partial_t A=\epsilon A-|A|^2A-g_{\mathbf{kq}}|B|^2A,$$

$$\tau_{\mathbf{k}}\partial_t B=(\epsilon-\epsilon_0(\mathbf{k}))B-\tau_{\mathbf{k}}|B|^2B-g_{\mathbf{qk}}|A|^2B, \qquad (4)$$

where we have assumed, since $\mathbf{q}$ is close to $\mathbf{q}_c$, $\epsilon_0(\mathbf{q})\simeq 0$ and $\tau_{\mathbf{q}}\simeq 1$. The main coupling coefficient is

$$g_{\mathbf{qk}}=-\tau_{\mathbf{k}}\langle U_1(\mathbf{k}),T_{\mathbf{qk}}\rangle$$

where

$$\begin{aligned}T_{\mathbf{qk}}=&N_2(V_1(-\mathbf{q}),V_2(\mathbf{q},\mathbf{k}))+\text{perm}\\&+N_2(V_1(\mathbf{q}),V_2(-\mathbf{q},\mathbf{k}))+\text{perm}\\&+N_2(V_2(\mathbf{q},-\mathbf{q}),V_1(\mathbf{k}))+\text{perm}\\&+N_3(V_1(\mathbf{q}),V_1(-\mathbf{q}),V_1(\mathbf{k}))+\text{perm}.\end{aligned} \qquad (5)$$

The OR $\mathbf{q}$ solution of Eq. (4), $A=\sqrt{\epsilon},B=0$, loses stability against a BV $(\mathbf{q};\mathbf{k})$ if the secondary growth rate

$$\tau_{\mathbf{k}}\sigma'=\epsilon(1-g_{\mathbf{qk}})-\epsilon_0(\mathbf{k}) \qquad (6)$$

becomes positive, i.e., only if $g_{\mathbf{qk}}<1$, and for ϵ then larger than

$$\epsilon_V(\mathbf{q};\mathbf{k})=\frac{\epsilon_0(\mathbf{k})}{1-g_{\mathbf{qk}}}. \qquad (7)$$

The active mode $\mathbf{k}$ which minimizes this secondary threshold is the dual of $\mathbf{q}$. In the case of a primary supercritical instability towards the OR, as assumed here, one has $g_{\mathbf{qk}}\rightarrow 2$ when $\mathbf{k}\rightarrow\mathbf{q}$. On the other hand, when $\mathbf{k}$ moves away from $\mathbf{q}$, $g_{\mathbf{qk}}$ can reach strongly negative values which favor the bimodal instability. Figure 1 shows a typical result for $g_{\mathbf{qk}}$ in thermoconvection, calculated with the parameters for the liquid crystal 5CB at 27 °C [3], that we used in the experiments [8]. The position of the dual mode is selected by a balance between the nonlinear interaction factor $(1-g_{\mathbf{qk}})^{-1}$, which favors $\mathbf{k}$ far from $\mathbf{q}$, and the linear cost $\epsilon_0(\mathbf{k})$ of the mode $\mathbf{k}$, which favors $\mathbf{k}$ close to $\mathbf{q}$ (in particular, preferentially $|\mathbf{k}|\simeq|\mathbf{q}|$). The calculated dual mode, shown in Fig. 1, corresponds qualitatively to the one observed experimentally [8]. Indeed $\mathbf{k}$ is in the zag region, far away from $\mathbf{q}$, but with a smaller angle between $\mathbf{k}$ and $\hat{\mathbf{x}}$: here $(\mathbf{k},\hat{\mathbf{x}})\simeq 25°$ instead of $\simeq 55°$ in the experiments. This discrepancy could certainly be reduced by using more sophisticated numerical methods, but we will see that it does not affect our results concerning the mechanisms of the BV instability.

The main advantage of this WNL approach is that the secondary growth rate of the bimodal instability Eq. (6) has an explicit ϵ dependence, governed by $g_{\mathbf{qk}}$ only. The origin of the BV instability can then be found by extracting from Eq. (5) the terms responsible for the strongly negative values of $g_{\mathbf{qk}}$ in the dual region. We now get into the details of this study for thermoconvection. There, the dimensionless local state vector of the fluid is $V=(\phi,n_y,n_z,f,g)$, where $\phi=$ temperature modulation, $f,g=$ velocity potentials [10]. With

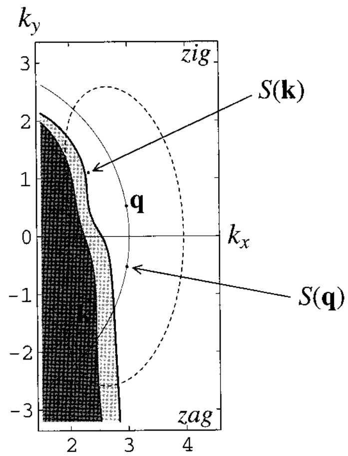

FIG. 1. Results of the stability analysis of a $\mathbf{q}$ zig against short-wavelength perturbation modes $\mathbf{k}$. Thin line: circle $|\mathbf{k}|=|\mathbf{q}|\simeq\pi$ in units of the inverse layer thickness. Dashed line: frontier of the active mode domain defined by $\sigma(\mathbf{k},\epsilon_V)>-0.15$ $(\epsilon_V=0.09)$. White region: $g_{\mathbf{qk}}>1\Leftrightarrow$ the mode $\mathbf{k}$ cannot destabilize the mode $\mathbf{q}$. Light gray region: $0<g_{\mathbf{qk}}<1\Leftrightarrow$ the mode $\mathbf{k}$, when linearly excited, can destabilize the mode $\mathbf{q}$ $[\epsilon_V(\mathbf{q};\mathbf{k})>\epsilon_0(\mathbf{k})]$. Deep gray region: $g_{\mathbf{qk}}<0\Leftrightarrow$ the mode $\mathbf{k}$ is nonlinearly excited by the mode $\mathbf{q}$ $[\epsilon_V(\mathbf{q};\mathbf{k})<\epsilon_0(\mathbf{k})]$. Under the application of $S:k_y\mapsto -k_y$, $g_{\mathbf{qk}}$ is asymmetric. It reaches strongly negative values only for active modes $\mathbf{k}$ zag, and thus the minimum of $\epsilon_V(\mathbf{q};\mathbf{k})$ (= bimodal varicose threshold ϵ_V) is obtained for a $\mathbf{k}$ zag.

the lowest-order Galerkin projection technique to expand the vertical dependence of the fields, the linear modes are approximated by:

$$V_1(\mathbf{q})=(\tilde{\phi}S_1(z),\tilde{n}_yS_2(z),\tilde{n}_zS_1(z),\tilde{f}C_1(z),\tilde{g}S_2(z))\ e^{i\mathbf{q}\cdot\mathbf{r}}$$

There S_n are sine and C_n are Chandrasekhar functions, of parity $(-1)^{n+1}$ under the inversion $z\mapsto -z$, $z=0$ being the mid-plane of the layer. The second harmonics have opposite vertical symmetry under this inversion. For instance, the homogeneous second harmonic reads:

$$\begin{aligned}V_2(\mathbf{q},\mathbf{q}')=(&\tilde{\phi}^HS_2(z),\tilde{n}_y^HS_1(z),\tilde{n}_z^HS_2(z),\\&\times\tilde{f}^HC_2(z),\tilde{g}^HS_1(z))\ e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{r}}\end{aligned} \qquad (8)$$

with $\mathbf{q}'=-\mathbf{q}$. The ten components of $V_1(\mathbf{q})$ and $V_1(\mathbf{k})$ are calculated as eigenvectors of the "matrices" $D^{-1}L_0$ and $D^{-1}L_{\epsilon_0(\mathbf{k})}$; the 15 components of $V_2(\mathbf{q},-\mathbf{q}),V_2(\mathbf{q},\mathbf{k})$ and $V_2(\mathbf{k},-\mathbf{q})$ are then obtained by the nonlinear adiabatic formulas (3). $g_{\mathbf{qk}}$ is then calculated as the scalar product of Eq. (5):

$$\begin{aligned}g_{\mathbf{qk}}&=-\tau_{\mathbf{k}}\langle U,T\rangle\\&=-\tau_{\mathbf{k}}(U_\phi T_\phi+U_{n_y}T_{n_y}+U_{n_z}T_{n_z}+U_fT_f+U_gT_g).\end{aligned} \qquad (9)$$

For each of the fields $\phi,n_y,\ldots$, each quadratic and cubic term in the corresponding basic equation gives six contributions according to Eq. (5), and as a result $g_{\mathbf{qk}}$ is a sum of more than 3000 terms. The contributions of each of these terms are shown on top of Fig. 2. The n_y contributions, like

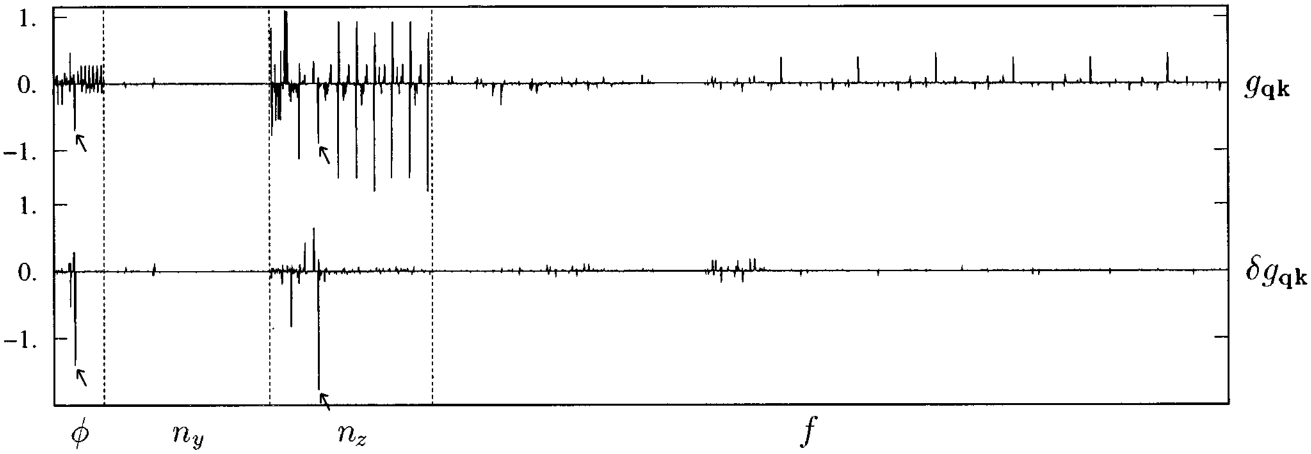

FIG. 2. Contributions of all nonlinear terms in the ϕ, n_y, n_z, and f basic equations to $g_{\mathbf{qk}}$, for the wave vectors $\mathbf{q}$ and $\mathbf{k}$ of Fig. 1. The first range of indexes, on the discrete horizontal axis, refers to the contributions due to T_ϕ in Eq. (9), the second range to the contributions due to T_{n_y}, etc. In each range the elementary terms are ordered according to Eq. (5). The asymmetry of $g_{\mathbf{qk}}$ between zag and zig perturbation wavevectors, $\delta g_{\mathbf{qk}} = g_{\mathbf{qk}} - g_{\mathbf{q}S(\mathbf{k})}$, is shown on the lower curve to be mainly due to two nonlinear coupling terms (arrows).

those of the g velocity potential (not shown), are all much smaller than the largest ones in the other fields contributions, but strong contributions due to T_ϕ, T_{n_z} or T_f exist. The complexity inherent in nonlinear physics, which implies the generation of harmonics, and so the interaction of a large number of modes via a large number of coupling terms, is manifest in Fig. 2. This complexity implies that a lot of mechanisms are, at small angles $(\mathbf{k}, \hat{\mathbf{x}})$, responsible for the bimodal secondary instability. Nevertheless, we will show that only few mechanisms control the asymmetry of the surface $g_{\mathbf{qk}}$ under the application of the reflection $S: k_y \mapsto -k_y$, or the values of $g_{\mathbf{qk}}$ at large angles $(\mathbf{k}, \hat{\mathbf{x}})$.

The asymmetry of the surface $g_{\mathbf{qk}}$ under all reflections (see Fig. 1) is specific to anisotropic systems. In isotropic convection for instance, the coupling coefficient between cross rolls $\mathbf{q}$ and $\mathbf{k}$ would be invariant under the reflection with respect to the plane $(\mathbf{q}, \hat{\mathbf{z}})$. In our case $g_{\mathbf{qk}}$ takes values more negative for $\mathbf{k}$ zag than for its symmetric $S(\mathbf{k})$ zig, i.e., $\delta g_{\mathbf{qk}} = g_{\mathbf{qk}} - g_{\mathbf{q}S(\mathbf{k})}$ is negative for $\mathbf{k}$ in the zag region, and that selects the dual in the zag region. In $\delta g_{\mathbf{qk}}$, many terms cancel, and thus (bottom of Fig. 2) two contributions dominate all the other ones, by at least 40% [11]. These contributions appear in T_ϕ and T_{n_z}, respectively, i.e., in the corresponding right-hand side of the basic equations (1) for these two fields. Since the light-hand side of Eq. (1) reads simply for these fields $\partial_t \phi = \cdots, \partial_t n_z = \cdots$, we call them ϕ and n_z source terms. They appear in the third line of Eq. (5), and couple the n_y component of $V_2(\mathbf{q}, -\mathbf{q})$ to some components of $V_1(\mathbf{k})$. The first contribution is due to the ϕ source term

$$-R\kappa_a \partial_y (n_y n_z) = -R\kappa_a \tilde{n}_y^H(\mathbf{q}) i k_y \tilde{n}_z(\mathbf{k}) S_1^2(z) e^{i\mathbf{k}\cdot\mathbf{r}} \quad (10)$$

from the anisotropic heat diffusion terms, R being the Rayleigh number and κ_a the anisotropy of heat diffusivity. The second contribution is due to the n_z source term

$$n_y \partial_y v_z = \tilde{n}_y^H(\mathbf{q}) i k_y \mathbf{k}^2 \tilde{f}(\mathbf{k}) S_1(z) C_1(z) e^{i\mathbf{k}\cdot\mathbf{r}} \quad (11)$$

from the α_2 viscous coupling terms (the director equation involves the anisotropic viscosities α_2 and α_3, but the α_3 contribution is here negligible). We will now elucidate in two steps the mechanism expressed by terms (10) and (11).

First, we focus on the n_y factor in terms (10) and (11). It is the n_y component of the second harmonic $V_2(\mathbf{q}, -\mathbf{q})$ (8), horizontally homogeneous and even in z. It thus indicates a *global rotation of the director in oblique rolls*. When one passes from the zig to the zag, i.e., applies $S: q_y \mapsto -q_y$, one has $\tilde{n}_y^H(\mathbf{q}) \mapsto -\tilde{n}_y^H(\mathbf{q})$: the rotation occurs only for OR, in symmetrical directions for the zig and the zag (Fig. 3). The main contributions to $\tilde{n}_y^H(\mathbf{q})$ are due to the quadratic n_y source terms $n_z \partial_z v_y - n_y \partial_x v_x$, from the α_2 viscous coupling terms, and $-v_z \partial_z n_y$ from the advection terms. Therefore, the rotation can be heuristically understood, after noticing that, because of the α_2 terms in the director evolution equation, $\hat{\mathbf{n}}$ reorients in order to avoid velocity gradients. In OR, this can be achieved by a rotation of $\hat{\mathbf{n}}$ towards the direction of the flow where there is no velocity gradient, i.e., the axis of the rolls.

Second, we analyze the resonances which can be induced by the terms (10) and (11). With our phase choice, $\tilde{\phi}(\mathbf{k})$,

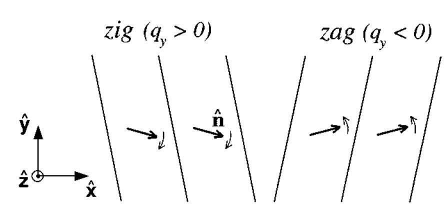

FIG. 3. Nonlinear reorientation of the director field $\hat{\mathbf{n}}$ in oblique rolls. The layer is seen from above; $\hat{\mathbf{x}}=$ anchoring direction $= \hat{\mathbf{n}}$ at rest. For 5CB at 27 °C, one should have in the midplane of the layer $n_y = \tilde{n}_y^H(\mathbf{q})\epsilon \simeq \pm 1.7\,\epsilon$, depending on the S symmetry of the oblique roll structure. This reorientation selects the S symmetry (zig or zag) of the secondary wave vector $\mathbf{k}$ in the bimodals, with the criterion $\tilde{n}_y^H(\mathbf{q}) k_y > 0$.

$\widetilde{f}(\mathbf{k})$, and $\widetilde{n}_z(\mathbf{k})/i$ are always positive real numbers. Therefore the ϕ source term (10) and the n_z source term (11) will excite the ϕ and n_z components of the mode $\mathbf{k}$ if $\widetilde{n}_y^H(\mathbf{q})k_y>0$. For $\mathbf{q}$ zig, since $\widetilde{n}_y^H(\mathbf{q})<0$, this occurs if $k_y<0$, i.e., selects the dual $\mathbf{k}$ in the zag region. Physically, terms (10) is a geometrical correction to the heat focusing, and terms (11) to the flow-induced orientation of the director. They express that, after a rotation of the director, ϕ and n_z modulations with a new wave vector oriented in the same direction will be preferentially enhanced.

The couplings of the convective velocity field with the director, and the focusing terms, being quite analogous in thermoconvection and electroconvection, these nonlinear mechanisms must also develop in electroconvection. This is confirmed by the systematic observation of the secondary transition OR → BV in the regime where the director distortion is the less damped: at low stabilizing magnetic field in thermoconvection, and at low frequency ω of the applied voltage in electroconvection [8,7].

These experiments have shown that the BV instability usually produces dual modes with a large angle $\arg\mathbf{k}=(\mathbf{k},\hat{\mathbf{x}})$: cf. the angle of 55° measured in [8], or, in electroconvection at low ω, the angle of 70° found recently in [7]. Since in both cases $\epsilon_0(\mathbf{k})$ reaches very high values for $\arg\mathbf{k}\rightarrow 90°$, this corresponds in our model [see the expression (7) of the BV threshold] to the fact that $g_{\mathbf{qk}}$ reaches there very negative values. Indeed, on the circle $|\mathbf{k}|=|\mathbf{q}|$, we have found that as soon as $\arg\mathbf{k}>40°$, Eq. (11) becomes the leading contribution to $g_{\mathbf{qk}}$, and that for $\arg\mathbf{k}\rightarrow 90°$, $g_{\mathbf{qk}}$ reaches, due to this contribution, values of the order of $-F=-$ the ratio of thermal to orientational diffusivity $\simeq -800$. The mechanism associated with Eq. (11) becomes thus for $\arg\mathbf{k}>40°$ the *unique leading mechanism towards the BV*.

In conclusion, we have developed a WNL model to describe the bimodal secondary instability in anisotropic extended systems. We have then shown how a careful study of the coupling coefficient leads to the nonlinear terms of the basic equations which control the bimodal instability. We have thus discovered that, in planar nematic convection, a reorientation of the director field in the horizontal plane is the key feature that excites the dual wave vector destabilizing the oblique rolls. We believe that our method to extract the physical mechanisms is applicable to all nonlinear systems in the low amplitude regime.

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[11] This appeared to be a very robust feature of $\delta g_{\mathbf{qk}}$: we obtained the same result with the parameters for MBBA used in [10], at null planar magnetic field and also for the bimodal predicted there at intermediate planar magnetic field ($h=34$).